Coherent Atomic Oscillations and Resonances between Coupled Bose-Einstein Condensates with Time-Dependent Trapping Potential

F.Kh. Abdullaev

Physical-Technical Institute, Uzbek Academy of Sciences,

700084, Tashkent-84, G.Mavlyanov str.,2-b, Uzbekistan

and

Instituto de Física Teorica, Universidade Estadual Paulista,

Rua Pamplona 145, 01405-900, São Paulo, Brasil

R.A. Kraenkel

Instituto de Física Teorica, Universidade Estadual Paulista,

Rua Pamplona 145,01405-900, São Paulo, Brasil

Abstract

We study the quantum coherent-tunneling between two Bose-Einstein condensates separated through an oscillating trap potential. The cases of slowly and rapidly varying in time trap potential are considered. In the case of a slowly varying trap we study the nonlinear resonances and chaos in the oscillations of the relative atomic population. Using the Melnikov function approach, we find the conditions for chaotic macroscopic quantum-tunneling phenomena to exists. Criteria for the onset of chaos are also given. We find the values of frequency and modulation amplitude which lead to chaos on oscillations in the relative population, for any given damping and the nonlinear atomic interaction. In the case of a rapidly varying trap we use the multiscale expansion method in the parameter $\varepsilon = 1/\Omega$, where Ω is the fre-

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quency of modulations and we derive the averaged system of equations for the modes. The analysis of this system shows that new macroscopic quantum self trapping regions, in comparison with the constant trap case, exist.

I. INTRODUCTION

It has recently been shown that there exists a macroscopic quantum phase difference in processes connected with atomic waves. Namely, this effect is observed between two tunnel-coupled Bose-Einstein condensates [1]- [3]. The coupling occurs due to a trap potential which has the form of a double-well potential. The theoretical investigation shows that here is possibly an interesting phenomenon of periodic oscillations of the atomic population between condensates and quantum self-trapping of population in the dependence of the relative phase between condensates [4]- [8]. Analogous phenomena which have been studied are the ac Josephson effect [9] and the periodic exchange of power and switching of electromagnetic waves between cores in nonlinear optical couplers [10].

In particular, there is a direct analogy between the tunneling phenomena in two prolongated Bose-Einstein condensates and two tunnel-coupled single mode optical fibers. In this optical analogy, the role of the chemical potential is played the propagation constant, and to the nonlinear interaction between atoms is analogous to the Kerr nonlinearity of optical media. The tunnnel-coupling arising from the overlaps of the electromagnetic fields outside of dielectric cilinders (fibers) [11–13] exactly corresponds to the tunnel coupling between two Bose-Einstein condensates due to the overlaps of the wavefunctions.

In this context , it is natural to investigate the influence of a time varying coupling on the quantum coherent tunneling process. In the nonlinear optical coupler analogy this corresponds to the variation in the longitudinal direction coupling [14,15]. As was shown recently, the variation in time of the trap potential can lead to resonant oscillations of the Bose-Einstein condensate [16,17]. The numerical solution of the Gross-Pitaevsky equation shows that the rapid variation in time of the trap potential can lead to a bifurcation of the effective (averaged) form of the trap potential and consequently to splitting of the BEC [18].

From what we have said above, we could wait for new phenomena in the process of quantum coherent atomic tunneling (QCAT) process too. Recalling the analogy with Josephson

effects in coupled junctions, we note that in the last case it is very difficult to produce such kind of effects, the difficulty coming from the fact that it is hard to implement the time varying overlap properties of junctions.

Thus, in this article we will study the atomic tunneling between two tunnel-coupled BEC's in a double-well *time dependent* trap. Namely, in Section II we will study the nonlinear resonance phenomena in the interference and, in particular, the stationary regimes, with damping taked into account. The phase and population damping are considered as well. The phase locked states are interesting because the suppression in these states of the fluctuations of the relative phase between two condensates. The regions of regular and chaotic oscillations of the relative atomic population are calculated in Section III, using the Melnikov function approach. In Section IV the macroscopic quantum interference in two overlapped condensates with rapidly varying traps is investigated. We derive the system of averaged equations for coupled modes and define in Section V new regimes of parameters for the macroscopic quamtum self-trapping (MQST).

II. FORMULATION OF PROBLEM. SLOWLY VARYING IN TIME TRAP

The problem of Bose-Einstein Condensates in a two double-well time dependent trap can be described by the following model

$$
\begin{aligned}
ih\frac{\partial \psi_1}{\partial t} &= (E_1(t) + \alpha_1|\psi_1|^2)\psi_1 - K(t)\psi_2 \\
ih\frac{\partial \psi_2}{\partial t} &= (E_2(t) + \alpha_2\psi_2|^2)\psi_2 - K(t)\psi_1,
\end{aligned}
\tag{1}
$$

where the parameters $E_i, \alpha_i, K(t)$ are defined by the overlaps integrals of the time dependent Gross-Pitaevsky eigenfunctions as given in [4]. α_i are parameters of nonlinear interactions between atoms and $\alpha_i \sim g_0 = 4\pi h^2 a/m$, a is the atomic scattering length. The attractive case can be obtained analogously, using symmetries of the equation This system of equations is valid in the approximation of a weak link between condensates. Comparison with the numerical solution of GPE shows that the system (1) is a reasonable approximation for

$z \leq 0.5 - 0.6$. For the strongly overlapped condensates the system should be modified. For a periodically varying $K(t) = K_0 + K_1 \sin(\Omega t)$ we have limitations on the parameters K_1 and Ω for which the two-mode approximation is valid. The small deformation of bound states requires $K_1 \ll K_0$. To avoid the resonances of the modulations with normal modes of the spherical well we should require that the difference of energy between the ground state and the first normal mode be larger than the modulation energy, i.e. $\delta E \gg \hbar \Omega$. Taking into account that $\delta E \sim \hbar \omega_0$, [7], where ω_0 is the harmonic oscillator frequency, we find the condition $\Omega \ll \omega_0$. Taking into account that $\omega_0 \gg \omega_L = 2K_0$ – the frequency of the linear oscillations of the atomic population, defined by the tunneling frequency, we conclude that by a proper choice of parameters we can have the resonant case $\Omega \simeq \omega_L$ or the rapidly varying modulation $\Omega \gg \omega_L$.

Introducing new variables $u_i = \sqrt{N_i} \exp(i\theta_i), z = (N_1 - N_2)/N_T, N_T = N_1 + N_2, \psi = \theta_1 - \theta_2$, where N_i, and θ_i are respectively the number of atoms and phases in i-th trap, we get the following system

$$z_t = -2K(t)\sqrt{1 - z^2} sin\Phi - \eta \Phi_t, \tag{2}$$

$$\Phi_t = \nu \Lambda z + \frac{2K(t)z}{\sqrt{1 - z^2}} \cos \Phi + \Delta E(t), \tag{3}$$

where $\Lambda = (\alpha_1 + \alpha_2)N_T$, and $\nu = \pm 1$ for the positive and negative atomic scattering length respectively. In what follows, we will consider the case $\nu = 1$, $(a_s > 0)$, if not stated otherwise. We have included in Eq(2) the damping term $\eta \phi_t(t)$. It appears if we take into account a noncoherent dissipative current of normal-state atoms, proportional to the chemical potential difference $\Delta \mu$. For the other type of overlapping condensate there might be another type of damping. For example, for the two interacting condensates with different hyperfine levels in a single harmonic trap the damping has the form $\eta z(t)$.

The Hamiltonian of the unperturbed system $(K(t) = \text{const}, \Delta E = \text{const}, \eta = 0)$ is

$$H = \frac{\Lambda z^2}{2} - 2K\sqrt{1 - z^2} \cos(\Phi) + \Delta E z \quad . \tag{4}$$

Let us consider the case of periodic modulations of the tunnel coupling parameter, when $K(t) = K_0 + K_1 \sin \Omega t$ and the time-periodic energy difference is $\Delta E(t) = \Delta E + \Delta E_1 \sin(\Omega t)$.

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It is useful to introduce the parameter $\Delta = \Phi - \Omega t$, which is a slow-varying function of the time in a comparison with the period Ω. The averaged system of equations can be found by averaging over the period of fast oscillations $2\pi/\Omega$. Doing so, we get the system

$$\bar{z}_t = -K_1 \cos \bar{\Delta} \sqrt{1 - \bar{z}^2} - \eta \Delta_t - \eta \Omega, \tag{5}$$

$$\bar{\Delta}_t = -\Omega + \Delta E + \Lambda z - \frac{K_1 z}{\sqrt{1 - \bar{z}^2}} \sin \bar{\Delta}. \tag{6}$$

Let us now look for the fixed points of Eq.(5). Introduce first the notation $\alpha = \Omega - \Delta E$. From the system (5) we find the following equation for the fixed points

$$z_c = \sqrt{1 - \frac{\eta^2 \Omega^2}{K_1^2 \cos \Delta_c^2}}, \tag{7}$$

$$\alpha = z_c (\lambda - K_1^2 \frac{\sin 2\Delta_c}{2\eta\Omega}). \tag{8}$$

For small $z_c, z_c^2 << 1$ we can find the explicit solution

$$\Delta_c = \arccos(\frac{\eta\Omega}{K_1}),$$
$$z_c \approx \frac{\Omega - \Delta E}{\Lambda - \sqrt{K_1^2 - \eta^2 \Omega^2}}. \tag{9}$$

The existence of a stationary solution is connected with the circumstance that the damping of the oscillation in quantum tunneling is compensated by the periodic variation of the trap. In reference [6] the influence of the damping on the macroscopic quantum interference has been investigated numerically and degradation of the interference due to damping has been analyzed. The predicted stationary solutions, where these effects are compensated by the variable driving, are indeed phase - locked solutions. Thus we can think that the fluctuations of the relative phase will be suppressed in such states. It is an important prediction since BEC's have phase fluctuations and it is difficult to maintain the constant relative phase of condensates. This phenomenon can be useful to develop the phase standard for the Bose-Einstein condensate- a problem attracting considerable attention recently [19].

We can estimate the value of fixed points for an experimental situation. For example, when $\Omega = 0.7, \eta = 0.1, K_1 = 0.2, \Lambda = 2.5$, we obtain the fixed points $z_{1c} = 0.2599, z_{2c} =$

0.3037. Stability analysis shows that only the first point is stable and corresponds to the global attractor. The results of the numerical simulations of the initial system for the relative population $z(t)$ are presented in Fig.1. It shows that the agreement between theory and numerical simulations is good.

The same system of equations also describes the nonlinear Josephson type oscillations in the relative population of a driven , two-component Bose-Einstein condensate. The two component condensates where two different hyperfine states can be populated and confined in the same trap. A weak driving field couples two internal states and as result the oscillations in the relative population will occur [20]. In this case the damping term has the form $\Gamma = -\eta z$. Repeating the above mentioned procedure we obtain the averaged system

$$\bar{z}_t = -K_1\sqrt{1-\bar{z}^2}\cos\bar{\Delta} - \eta\bar{z},$$
$$\bar{\Delta}_t = \Lambda\bar{z} + \Delta E - \Omega - \frac{K_1\bar{z}}{\sqrt{1-\bar{z}^2}}\sin\bar{\Delta}. \tag{10}$$

In the model [20], ΔE corrsponds to $\mu_2 - \mu_1 - \kappa$, where μ_i are chemical potentials and κ is frequency detuning $\kappa = \omega_d - \omega_0$, ω_0 is the separation in frequency of two hyperfine states, ω_0 is the frequency of drive. The fixed points are:

$$\bar{\Delta}_c = \arccos\left(\frac{\eta z_c}{K_1\sqrt{1-z_c^2}}\right),$$
$$\Lambda z_c - \alpha - \frac{K_1 z_c}{\sqrt{1-z_c^2}}(1 - \frac{\eta^2 z_c^2}{K_1^2(1-z_c^2)})^{1/2}. \tag{11}$$

Here $\alpha = \Omega - \Delta E$. For the case of small $z_c^2 << 1$ we have the estimate for z_c

$$z_c = \frac{\Delta E - \Omega}{\Lambda - K_1} + \frac{(\Delta E - \Omega)^3(K_1^2 - \eta^2)}{2K_1(\Lambda - K_1)^4}. \tag{12}$$

The typical values of parameters are:$\Omega = 0.7, \mu_2 - \mu_1 - \kappa = \Delta E = 1, \eta = 0.1, K_1 = 0.2, \Lambda = 4$, and the fixed points are $z_{3c} = -0.111, z_{4c} = -0.13$. The analysis shows that the first fixed point is stable and second is unstable.

In Fig.2. we present the plot of solution of the full system for the damping term ηz with the above mentioned set of parameters. All solutions decay to the first fixed point.

III. CHAOTIC OSCILLATIONS IN RELATIVE ATOMIC POPULATION

In this section we consider the interference in the system which in some limit is mathematically equivalent to the motion of a particle in a double-well potential under parametric periodic perturbations. The particle' motion is of course unharmonic and the nonlinear resonances between oscillations of a particle and oscillations of the trap potential are possible. One of consequences of this is the possible appearence of chaotic dynamics in such systems. We will now study the conditions for the appearence of the chaotic dynamics in atomic tunneling.

To proceed it will be useful first to consider some aspects of the unperturbed system, that is, when the coupling is constant. In this case the system (2) is equivalent to the quartic (Duffing) oscillator [6,23].

We can write the equation for $z(t)$ as

$$z_{tt} = -\frac{\partial V}{\partial z} \quad , \tag{13}$$

where the potential $V(z)$ is given by

$$V(z) = z^2(a + bz^2), a = 2K_0^2 - \frac{\Lambda H}{2}, \; b = \frac{\Lambda^2}{8}. \tag{14}$$

and H is the Hamiltonian. The total energy of the effective particle is

$$E_0 = \frac{z_t^2}{2} + V(z) = 2K^2 - H^2/2 \quad .$$

When $E_0 > 0$ we have the oscillating regime with $< z(t) >= 0$ (nonlinear Rabi oscillations). This case corresponds to the periodic flux of atoms from one BEC to other. When $E_0 < 0$ the motion of effective particle is confined in the one of the wells. This case corresponds to the localization of atomic population in one of the condensates - the so called macroscopic quantum self-trapping (MQST). The case $E_0 = 0$, i.e. $H = 2K$, corresponds to the separatrix solution separating these two regimes. It is interesting to investigate the dynamics of atomic population, when the inital parameters $z(0), \Phi(0)$ are close to the separatrix of the

unperturbed systsem. From the general theory of nonlinear driven oscillations [21] we can expect the appearence of *chaotic macroscopic tunneling phenomena.*

The separatrix solutions for the right hand side well(condensate) are:

$$z_s(t) = \sqrt{\frac{a}{b}}\text{sech}(\sqrt{2a}t) \quad,$$

$$\sin^2(\Phi_s) = \frac{a^2\text{sech}^2(\sqrt{2a}t)\tanh^2(\sqrt{2a}t)}{2K_0^2(b - a\text{sech}^2(\sqrt{2a}t))} \quad, \tag{15}$$

where $H = H_s = 2K$.

The motion near the separatrix is very sensitive to the change of the initial condition and a stochastic layer appeas, leading to the chaotic dynamics of the solutions of the system (2). The Melnikov function method is effective to find the regions of chaotic behavior [21,22]. The system (2) may written in the form $\mathbf{r}_t = \mathbf{f} + \epsilon\mathbf{g}$ and has a hyperbolic fixed point. According to this method, we need to calculate the Melnikov function $M(t_0)$

$$M(t_0) = \int_{-\infty}^{\infty} (f_{2s}g_{1s}(t, t_0) - f_{1s}g_{2s}(t, t_0))dt. \tag{16}$$

As is well known, the existence of zeros of $M(t_0)$) indicates the intersection of separatrices and the existence of homoclinical chaos. Substituting the expressions for f_i, g_i from the system (2) and calculating the integrals, we find the final expressions for the Melnikov function. We shall give the results for two possible damping terms.

A. The case of the damping in the form $\eta\Phi_t$

The integrations in (16) are cumbersome, and will be omitted. The final result for the Melnikov function is:

$$M_1(t_0) = -K_1F_1\cos(\Omega t_0) - F_2\eta \quad, \tag{17}$$

where

$$F_1 = \frac{\pi\Lambda\Omega^2}{4K_0b\sinh(\frac{\pi\Omega}{2\sqrt{2a}})},$$

$$F_2 = \frac{2\Lambda\sqrt{2a}}{\sqrt{ab - a^2}}(\Lambda - 2H_s)\arctan(\sqrt{\frac{a}{b - a}}) - \frac{\sqrt{2a}}{16(b - a)b^2}G_1 \quad, \tag{18}$$

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where

$$G_1 = [-2b^2(2H_s - \Lambda)(bH_s - 4a\Lambda + 3b\Lambda)]\frac{\arctan(a/\sqrt{ab - a^2})}{\sqrt{ab - a^2}} - $$
$$2\left[4a^2\Lambda^2 - 5ab\Lambda^2 + 2b^2(4H_s^2 - 4H_s\Lambda + 3\Lambda^2)\right] \quad . \tag{19}$$

The simple zero of $M(t_0)$ is absent if the condition

$$\eta > \frac{F_1}{F_2}, \tag{20}$$

is satisfied. In Fig.3 we plot this criterion in the (K_1, Ω) plane for $\eta = 0.1$. Regions below the curve correspond to regular oscillations and above to chaotic oscillations of $z(t)$. The criterion gives a good lower bound on the region of chaos in this plane. We verified this lower bound in (K_1, Ω) plane by numerical simulations and found a good agreement with the formulae (20).

When the damping coefficient is zero we have the expression for the width of stochastic layer near the separatrix as

$$D = \frac{\pi\Lambda K_1\Omega^2}{4K_0b\sinh(\frac{\pi\Omega}{2\sqrt{2a}})}. \tag{21}$$

In Fig.4 we plot the width of the stochastic layer as a function of the frequency for $K_1 = 0.2, \Lambda = 9.9, z(0) = 0.6, \Phi(0) = 0$. The maximum of the width is given by the solution of equation

$$\Omega\tanh(\frac{\pi\Omega}{2\sqrt{2a}}) = \frac{4\sqrt{2a}}{\pi}.$$

For this choice of parameters we have $\Omega_m = 1.89$. For $\Omega \geq 2, a \geq 1$ we get the estimate $\Omega_m \approx 4\sqrt{2a}/\pi$. It can be seen that, for the high frequencies $\Omega >> \sqrt{a}$, the stochastic layer is exponentially narrow and motion is regular. The analysis of this case will be performed in detail in the next section. In Fig.5 we plot the typical oscillations in the relative atomic population for $\Lambda = 9.9, K_0 = 1, K_1 = 0.2, z(0) = 0.6, \Phi(0) = 0$ for $\Omega = 1.5$. It turns out that for Ω between 1.5 and 3.22 we have chaotic oscillations, and for small or large Ω, regular oscillations. In Fig.6 the phase portrait is given. In Fig.7 the influence of the damping on chaos is illustrated. Parameters are the same as in Fig.5 and $\eta = 0.15$. In this case the oscillations become regular.

B. The case of the damping term in the form $\eta z(t)$.

The calculation of the integrals which give the Melnikov function is again cumbersome, and one find the following result:

$$M(t_0) = F_1 \cos(\Omega t_0) + \eta F_3, \tag{22}$$

where

$$F_3 = \frac{\Lambda - 2H_s}{\sqrt{2(b-a)}} \arctan\left(\sqrt{\frac{a}{b-a}}\right) + \frac{3\Lambda\sqrt{a}}{2\sqrt{2}b}. \tag{23}$$

The zeros of $M(t_0)$ are absent when the condition

$$\eta > \frac{F_1}{F_3}, \tag{24}$$

is satisfied.

If we consider for example the case when $K_1 = 0.2, \Omega = 0.7, \Lambda = 2.5, z(0) = 0.6, \Phi(0) = \pi$, the damping constant should be $\eta = 0.15$. In Figs.(7,8) the chaotic and periodic oscillations of the relative population in the dependence of the frequency for fixed $z(0), \Phi(0)$ are plotted.

IV. RAPIDLY VARYING TRAP POTENTIAL

In the rapidly varying case $K = K(\frac{t}{\epsilon})$ we can apply he averaging techniques based on the multiscale expansions [24,25]. Let us search the solution in the form of the sum of the slowly varying components U, V and rapidly varying corrections u_i, v_i

$$\psi_1(t) = U(t_k) + \epsilon u_1(\zeta) + \epsilon^2 u_2(\zeta) + ...$$

$$\psi_2(t) = V(t_k) + \epsilon v_2(\zeta) + \epsilon^2 v_2(\zeta) + ... \tag{25}$$

where $\zeta = t/\epsilon, \epsilon << 1,$, $t_k = \epsilon^k t$ are slow times. The averages over a period of rapid oscillations are $< u_i >=< v_i >= 0$. The functions u_i, v_i are assumed to be functions of the

rapid time ζ and the slow varying functions U, V. The derivatives can be written in the form

$$\frac{dU}{dt} = \frac{dU}{dt_0} + \epsilon \frac{dU}{dt_1} + \dots \tag{26}$$

and analogously for the derivative of V.

Below, we will assume $E_1 = E_2$,. Then the time dependence in $E(t)$ can be removed by a simple field transformation $u(v) = u(v) \exp((i/h) \int E(t')dt')$. In the zeroth order in ϵ we get the equations

$$ih(\frac{\partial u_0}{\partial t_0} + \frac{\partial u_1}{\partial \zeta}) = Eu_0 - K(\zeta)v_0 + \alpha |u_0|^2 u_0 \tag{27}$$

$$ih(\frac{\partial v_0}{\partial t_0} + \frac{\partial v_1}{\partial \zeta}) = Ev_0 - K(\zeta)u_0 + \alpha |v_0|^2 v_0 \tag{28}$$

Requiring the exclusion of the secular in ζ terms, we find the corrections u_1, v_1

$$u_1 = \frac{i}{h}v_0(\mu_1 - <\mu_1>) \tag{29}$$

$$v_1 = \frac{i}{h}u_0(\mu_1 - <\mu_1>) \tag{30}$$

where $\mu_1 = \int_0^\zeta (K(s) - <K>)ds$ and the system for u_0, v_0 becomes:

$$ih\frac{\partial u_0}{\partial v_0} = Eu_0 - <K>v_0 + \alpha |u_0|^2 u_0 \tag{31}$$

$$ih\frac{\partial v_0}{\partial t_0} = Ev_0 - <K>u_0 + \alpha |v_0|^2 v_0 \tag{32}$$

In the order ϵ we have

$$\frac{\partial u_0}{\partial t_1} = 0, \ \frac{\partial v_0}{\partial t_1} = 0 \quad , \tag{33}$$

and we obtain, for the second order corrections, u_2, v_2, the elimination of the secular terms, results in:

$$u_2 = -\frac{i}{h}(\mu_2 - <\mu_2>)[\frac{\partial v_0}{\partial t_0} + \frac{i}{h}v_0 + \frac{i\alpha}{h}[2|u_0|^2 v_0 + u_0^2 v_0^*]]$$
$$-[(\mu_1 - <\mu_1>)^2 - 2M]u_0 \quad , \tag{34}$$

$$v_2 = -\frac{i}{h}(\mu_2 - <\mu_2>)[\frac{\partial u_0}{\partial t_0} + \frac{i}{h}u_0 + \frac{i\alpha}{h}[2|v_0|^2 u_0 + v_0^2 u_0^*]]$$
$$-[(\mu_1 - <\mu_1>)^2 - 2M]v_0 \quad , \tag{35}$$

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where

$$\mu_2 = \int_0^\zeta (\mu_1 \zeta - < \mu_1 >)ds, \ M = (< \mu_1^2 > - < \mu_1 >^2) \quad .$$

¿From this order we get the equations:

$$\frac{\partial u_0}{\partial t_2} = -\frac{2iM\alpha}{h^3}[2|v_0|^2 u_0 - v_0^2 u_0^*] \quad , \tag{36}$$

$$\frac{\partial v_0}{\partial t_2} = -\frac{2i\alpha M}{h^3}[2|u_0|^2 v_0 - u_0^2 v_0^*] \quad . \tag{37}$$

Finally, we have in the next order, $O(\epsilon^2)$, the averaged system

$$i\hbar \frac{\partial u_0}{\partial t} - Eu_0 - \alpha|u_0|^2 u_0 = - < K > v_0 + 2\frac{\epsilon^2 \alpha M}{\hbar^2}[2|v_0|^2 u_0 - v_0^2 u_0^*] \tag{38}$$

$$i\hbar \frac{\partial u_0}{\partial t} - Ev_0 - \alpha|v_0|^2 v_0 = - < K > v_0 + 2\frac{\epsilon^2 \alpha M}{\hbar^2}[2|u_0|^2 v_0 - u_0^2 v_0^*]. \tag{39}$$

For periodic modulations of tunnel coupling we obtain $M = K_1^2/\Omega^2$.

This averaged system is the main result of this section. We see that, in comparison with the constant coupling case, new effects appear. First, the cross term appears, corresponding to a change of type in the effective nonlinearity in coupled BEC' dynamics. Second, there appears an additional *nonlinear* phase-sensitive coupling. This term leads to new minima in the effective potential picture. In the constant coupling case, the oscillations of $z(t)$ is described by the Duffing oscillator. Now we have a more complicated nonlinear oscillator, describing more complicated interference patterns.

V. ANALYSIS OF THE AVERAGED DYNAMICS

Using the variables introduced in the section II, we get the system

$$z_t = -2 < K > \sqrt{1 - z^2} \sin(\Phi) - \nu\delta(1 - z^2)\sin(2\Phi) \tag{40}$$

$$\Phi_t = \Lambda z + 2\frac{< K > z}{\sqrt{1 - z^2}}\cos(\Phi) - \nu\delta z\left[2 - \cos(2\Phi)\right]. \tag{41}$$

where we have defined $\delta = \epsilon^2 \alpha M/h^2$. This system has the following Hamiltonian

$$H = \frac{\Lambda_1 z^2}{2} - 2 < K > \sqrt{1 - z^2}\cos\Phi - \frac{\delta}{2}(1 - z^2)\cos 2\Phi, \tag{42}$$

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where $\Lambda_1 = \Lambda - 2\delta$.

When the parameter $\delta = 0$, we get back the system (2). The effective particle motion is that of the double nonrigid pendulum system.

Let us consider the different regimes of oscillations.

A. Linear regime

This is the case where $|z| << 1, |\Phi| << 1$. Then, from the system (40), it follows that that

$$z_{tt} = -2(< K > +\delta)(\Lambda + 2 < K > -\delta)z.$$

Then the frequency of linear oscillations of the number of atoms is

$$\omega_L^2 = -2(< K > +\delta)(\Lambda + 2 < K > -\delta) \quad .$$

Thus, the frequency of the linear oscillations grows with $\delta\omega^2 = \omega_L^2 - \omega_0^2 = 2\delta(< K > +\Lambda - \delta)$, in comparison with a nonmodulated case.

B. $z^2 << 1$, and Φ is not small.

The phase dynamics is described by the equation:

$$\Phi_{tt} + 2 < K > (\Lambda - 2\delta)\sin\Phi + (\delta\Lambda + 2 < K >^2 - 2\delta^2)\sin(2\Phi) +$$
$$2\delta < K > \sin(3\Phi) + \frac{\delta^2}{2}\sin(4\Phi) = 0. \tag{43}$$

At derivation of this equation we take into account that $\Phi_t \sim z$, so we neglect by terms $\sim z\Phi_t$. This equation is equivalent to a generalized pendulum motion whose effective potential is given

$$U(\Phi) = -2 < K > (\Lambda - 2\delta) - \frac{1}{2}(\Lambda\delta + 2 < K >^2 - 2\delta^2)\cos(2\Phi) - \frac{2\delta < K >}{3}\cos(3\Phi) - \frac{\delta^2}{8}\cos(4\Phi).$$
$$\tag{44}$$

The additional minima at $\Phi = \pm n\pi$ correspond to the trapping of oscillations in states witt locked phase.

VI. MACROSCOPIC QUANTUM SELF-TRAPPING.

This phenomenon consists in the self trapping of the atomic number in one of condensates, depending on the initial difference in atomic number and on the relative phase of condensates . This is connected with the phase sensitive *linear coupling*. The nonlinear guided wave optics analog of this phenomenon, first predicted in [4], is the switching of the power in the dual core couplers from one core to other, depending on the initial power or the phase difference. The additional coupling in the averaged system (40) is *nonlinear* and also phase sensitive. Thus the condition of localization of atomic population should modified in comparison with the one considered earlier.

Let us find the fixed points of the averaged system (40). The first group is for $\Phi_n = \pi n, n = 0, \pm 1, \pm 2, ...$ For each Φ_n there can exist one or three roots for z. One of the roots is $z_1 = 0$, two others are

$$z_{s1,2} = \pm \left(1 - \frac{4 < K >^2}{(\Lambda - \delta)^2}\right)^{1/2}.$$
(45)

Note that if $(\Lambda - \delta) < K >< 0$, then 3 roots exists for even n and one root $(z_1 = 0)$ exist for odd n. For $(\Lambda - \delta) < K >> 0$,one has the opposite situation, the critical value is $\Lambda_c > 2 < K > +\delta$.

One can show, however, linear stability analysis predicts that this set is unstable when $\delta > \Lambda/2$.

The second group of fixed points is given by:

$$\cos(\Phi) = -\frac{< K >}{\sqrt{1 - z^2}}.$$
(46)

Again, there are two possibilities:

i)$\Lambda = 3\delta$, then all line is stationary;

ii)$\Lambda \neq 3\delta$, then one has the following static point

$$\cos(\Phi) = -\frac{< K >}{\delta}, z = 0.$$
(47)

This root assumes that $| < K > /\delta| < 1$. The linear stability analysis shows that the fixed points are unstable when $\Lambda > 3\delta$.

Following [6], from the constraint coming from the energy conservation and the boundness of the tunneling energy, given by

$$H = \frac{\Lambda_1 z^2(0)}{2} - 2 << K >> \sqrt{1 - z^2(0)} \cos\left(\Phi(0)\right) - \frac{\delta}{2}(1 - z^2(0))\cos(2\Phi(0)) \geq 2 < K > -\frac{\delta}{2} \quad ,$$

(48)

we obtain the estimate for the critical value of Λ_c when the self-localization occurs. We find that:

$$\Lambda_c = \frac{4 < K > (\sqrt{1 - z^2(0)}\cos\Phi(0) + 1) + \delta((1 - z^2(0))\cos 2\Phi(0) - 1)}{z^2(0)} + 2\delta.$$

(49)

The phase portrait is plotted in Fig.(9,10,11) for initial phase difference $\Phi(0) = \pi$ and $\Lambda = 2.5$, for differnt $\delta = 0.2; 0.6; 0.8$ The regions corresponding to different dynamics of $z(t)$ with $< z > = 0$ and $< z > \neq 0$ are shown. The increasing of δ (i.e. K_1/Ω) leads to the distortion of the MQST regime, and for $\delta \sim 1$ to the nonlinear Rabi-like oscillations. The typical oscillations in time of the relative population for different values of δ and for fixed Λ are plotted in Fig.9. The influence of rapid modulations leads to the appearance of new minima in the oscillations. This can be described as a result of the overlap of two double well potentials. This conclusion is also confirmed by the results of numerical simulations of the GP equation for the *single* BEC under rapidly varying trap potential performed in [18]. In this work, it has been shown that the effect of rapid modulations is the appearance of a effective double well trap potential.

VII. CONCLUSION

In this paper we have studied the new effects coming the time variation of the trap potential, and from damping, on the nonlinear oscillations in the relative population behavior between two BEC's. For the slowly varying trap we predict synchronization of oscillations

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of the trap with oscillations of the relative population. We find the fixed points for both types of the damping terms occuring in the studies of two coupled condensates. Using the Melnikov approach we study the possibility the appearance of the chaotic oscillations in the tunneling phenomena between two coupled Bose-Einstein condensates. We calculate the width of the stochastic layer in phase space, in which the dynamics is chaotic. We find the lower bound on the region of chaos in (K_1, Ω) plane. We obtain the estimate for the damping coefficient when the chaos is suppressed. For the rapidly varying trap we use the multiscale method and derive the averaged equations for coupled modes describing the tunneling phenomena. We find the fixed points in this case describing the stationary states in two coupled BEC's. The expression for the critical value of the nonlinearity parameter , when the macroscopic quantum self-trapping occurs is derived. Our results show that there is interest to study the contribution of the quantum tunneling for this driven case and the investigation of resonances and chaos in oscillations of atomic poulation for the strongly overlapped condensates [27]. These problems require separate investigation.

VIII. ACKNOWLEDGMENTS

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FIGURES
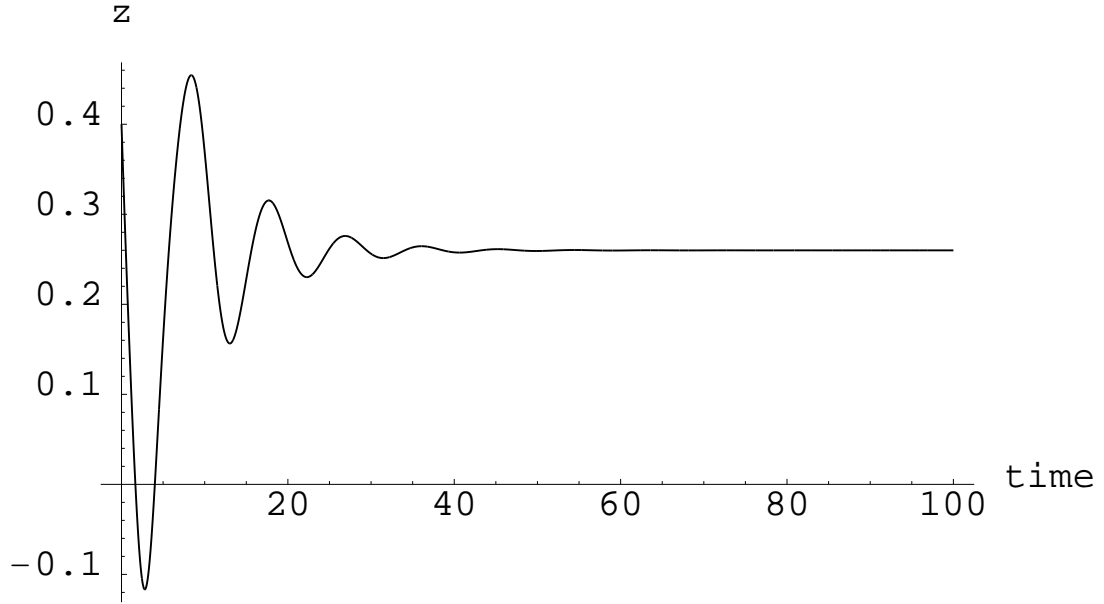
FIG. 1. The typical behavior of the solutions to Eq.(2) when the damping is taken in the form $\Gamma = \eta\phi_t$. This case corresp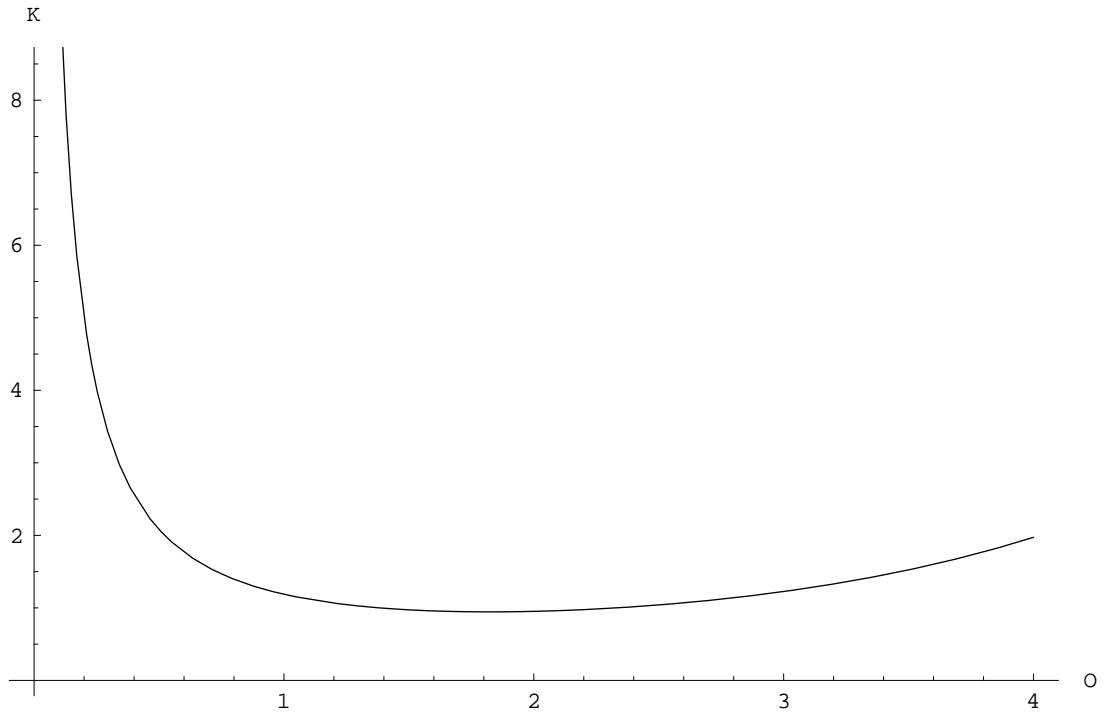onds to $\Omega = 0.7, \eta = 0.1, K_1 = 0.2, \Lambda = 2.5$. Theoretical value is $z_c = 0.2599$, the numerical is $z_c = 0.26$.
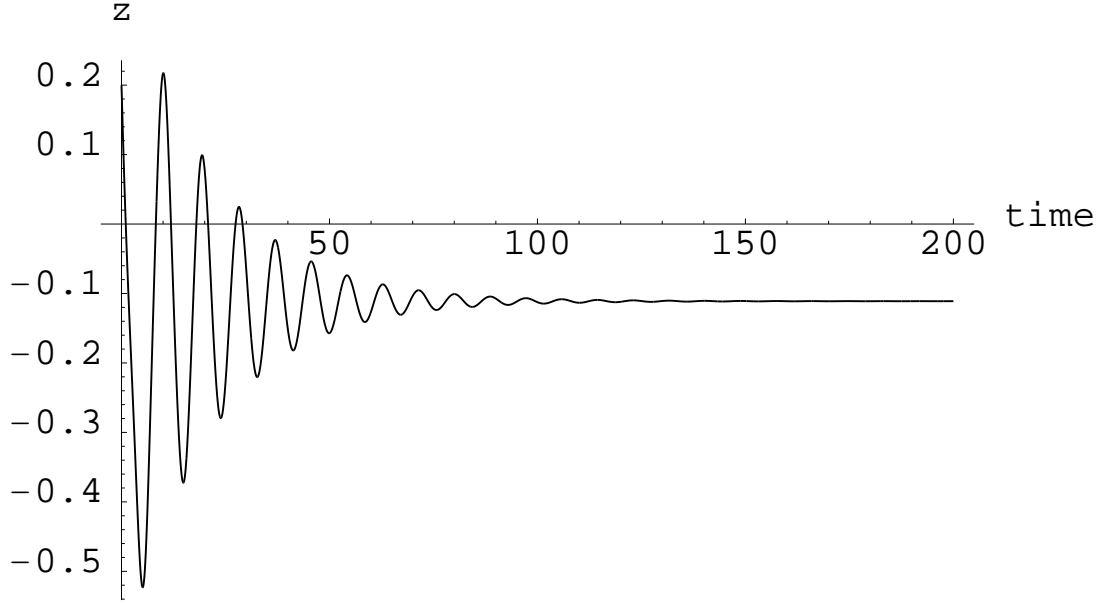
FIG. 2. The typical behavior of the solutions to Eq.(2) when the damping is taken in the form $\Gamma = \eta z$. This case corresponds to $\Omega = 0.7, \Delta E = 1, \eta = 0.1, K_1 = 0.2, \Lambda = 2.5$

FIG. 3. The regions of regular and chaotic oscillations in plane K_1, Ω from the Melnikov approach.
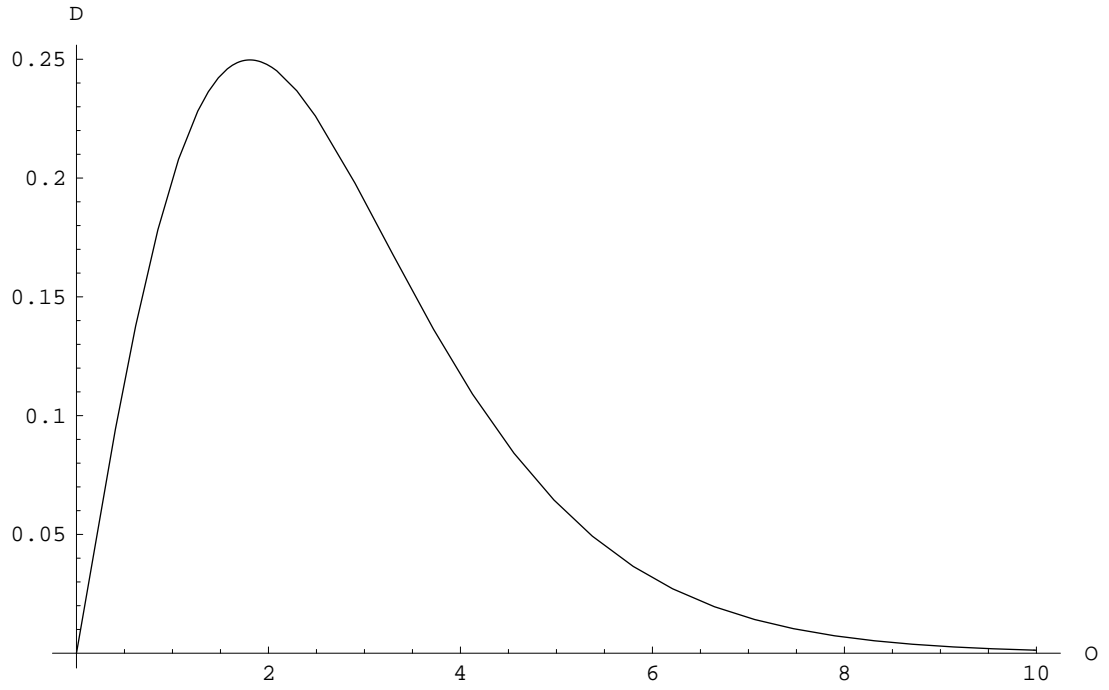
FIG. 4. The width of stochastic layer as a function of frequency Ω, when $\eta = 0, K_0 = 1, K_1 = 0.2, \Lambda = 9.9, z(0) = 0.6, \Phi(0) = 0$.
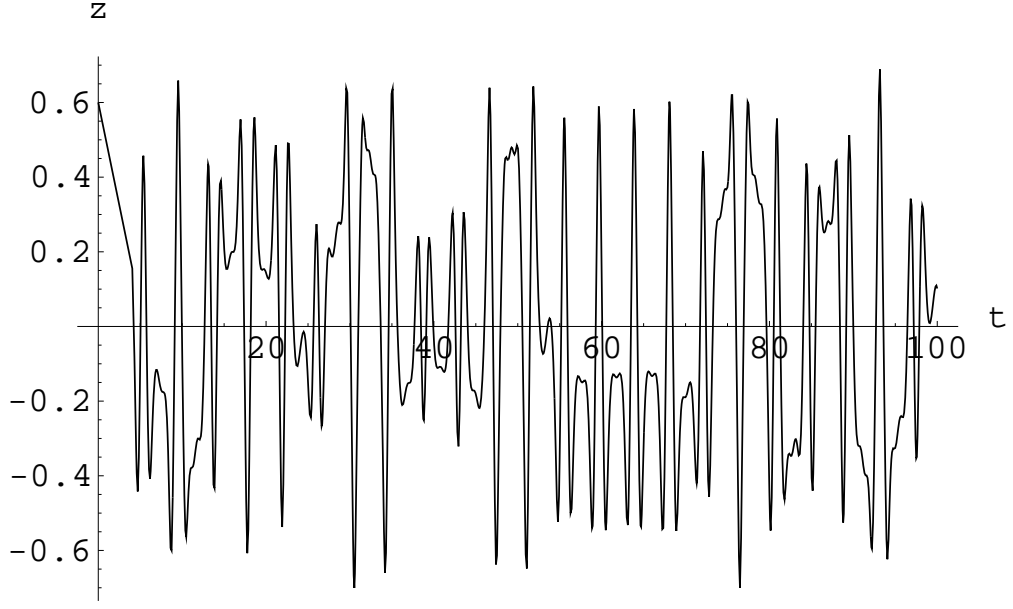
FIG. 5. The typical oscillations of the relative population $z(t)$ for $K_1 = 1.2, \Lambda = 9.9, z(0) = 0.6, \Phi_0 = 0, \Omega = 1.5$
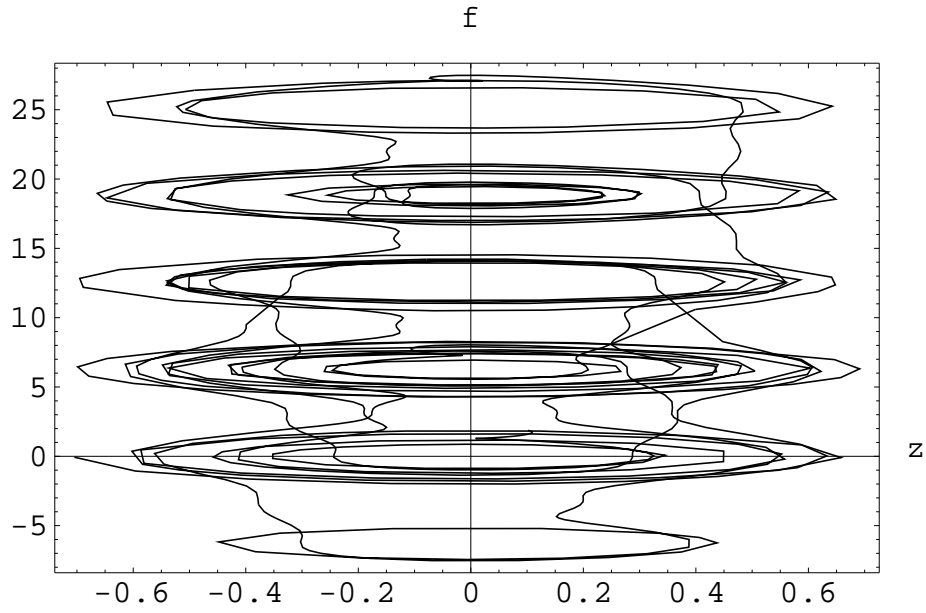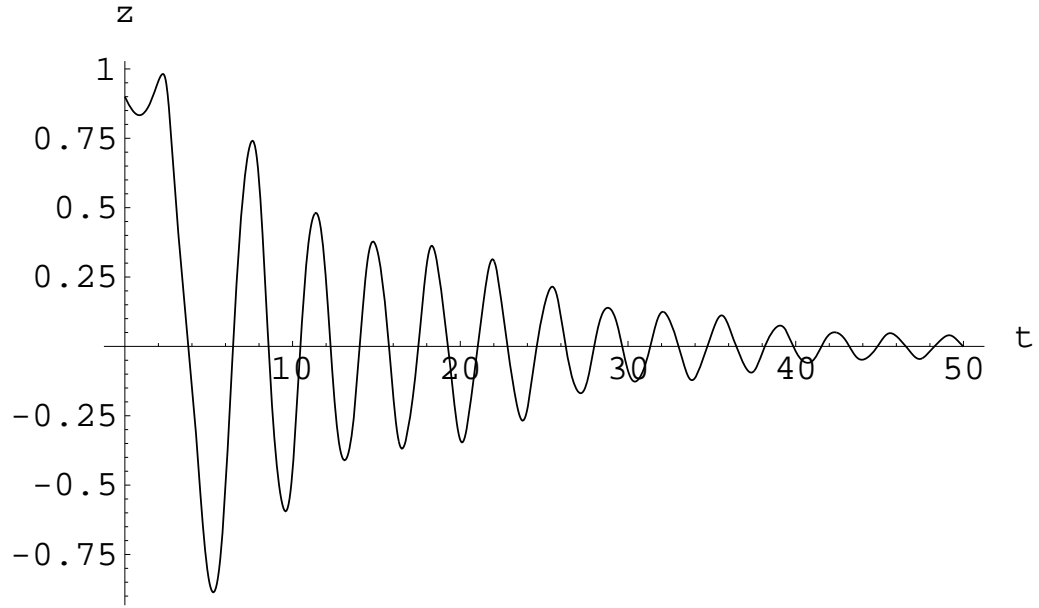
FIG. 6. The phase portrait in the (Φ, z)-plane for the same values of the parameters as in Fig(5).
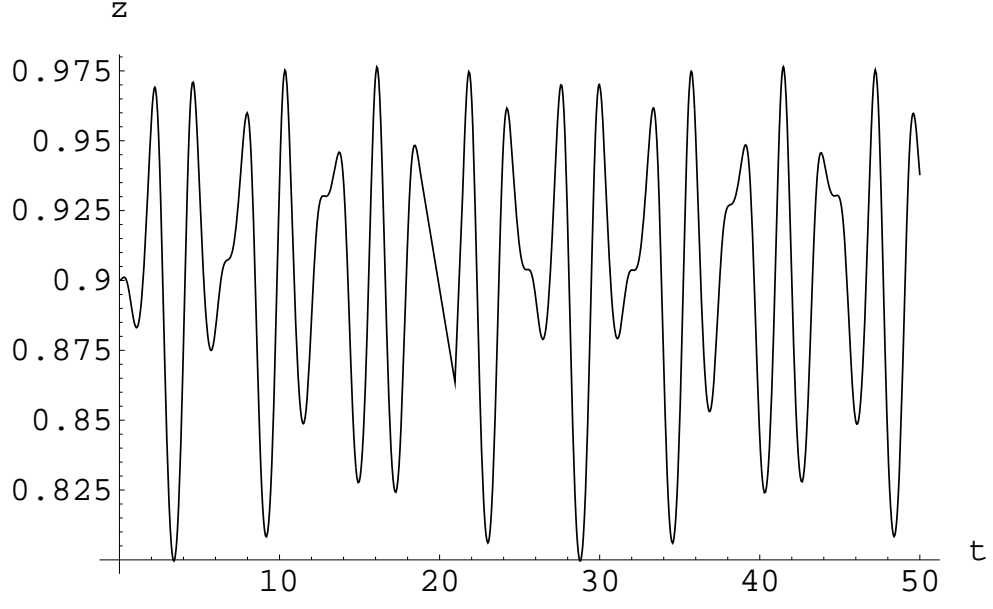
FIG. 7.   The typical oscillations of the relative population $z(t)$ for $K_1 = 0.2, \Lambda = 2.5, z(0) = 0.9, \Phi_0 = \pi, \Omega = 3.22$

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FIG. 8. The typical oscillations of $z(t)$ for the same set of parameters as in Fig.(7), but with $\Gamma = -0.15z$
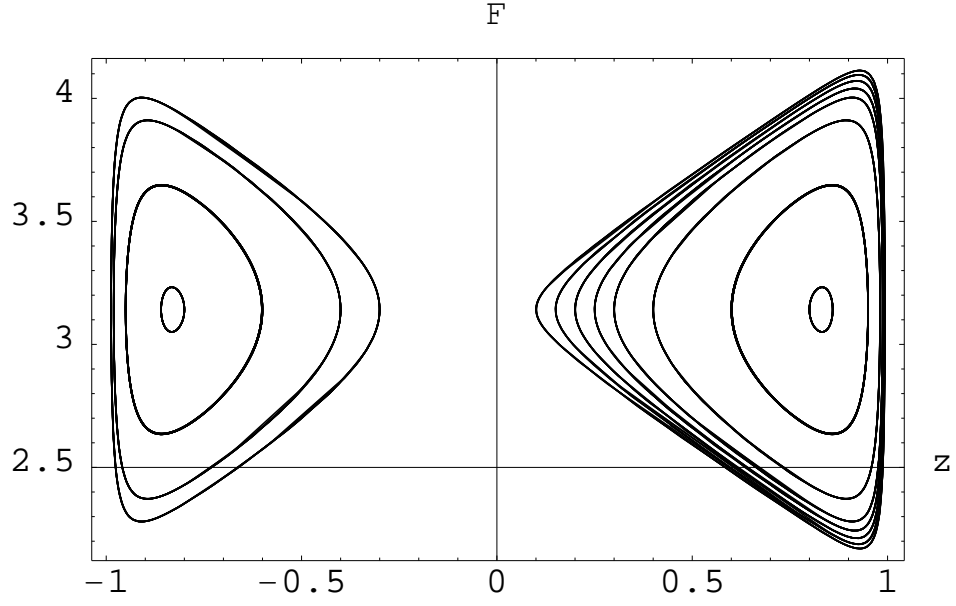
FIG. 9. The phase portrait in a plane z, ϕ of the averaged system of equations (40) for $\Phi(0) = \pi, \Lambda = 2.0, \delta = 0.2$
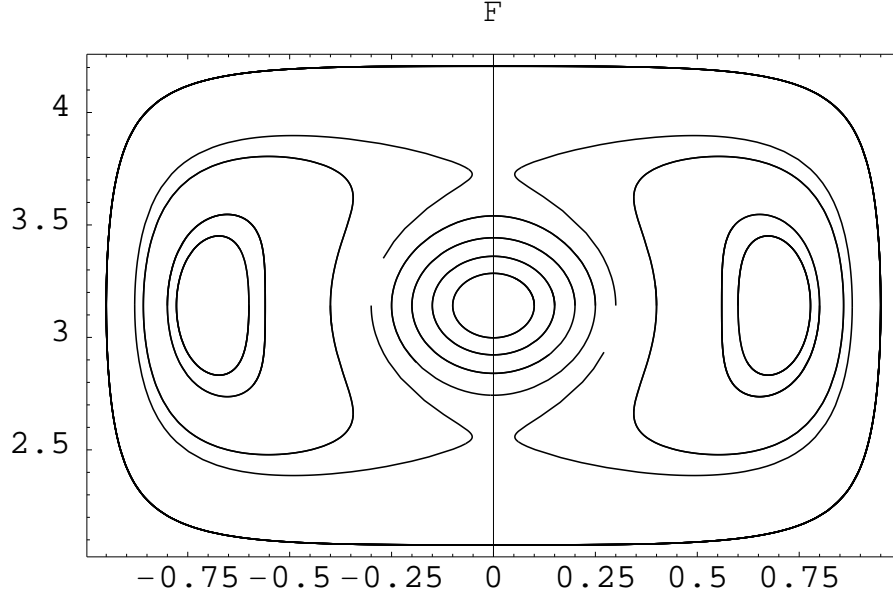
FIG. 10. The phase portrait in a plane z, ϕ of the averaged system of equations (40) for $\Phi(0) = \pi, \Lambda = 2.0, \delta = 0.6$
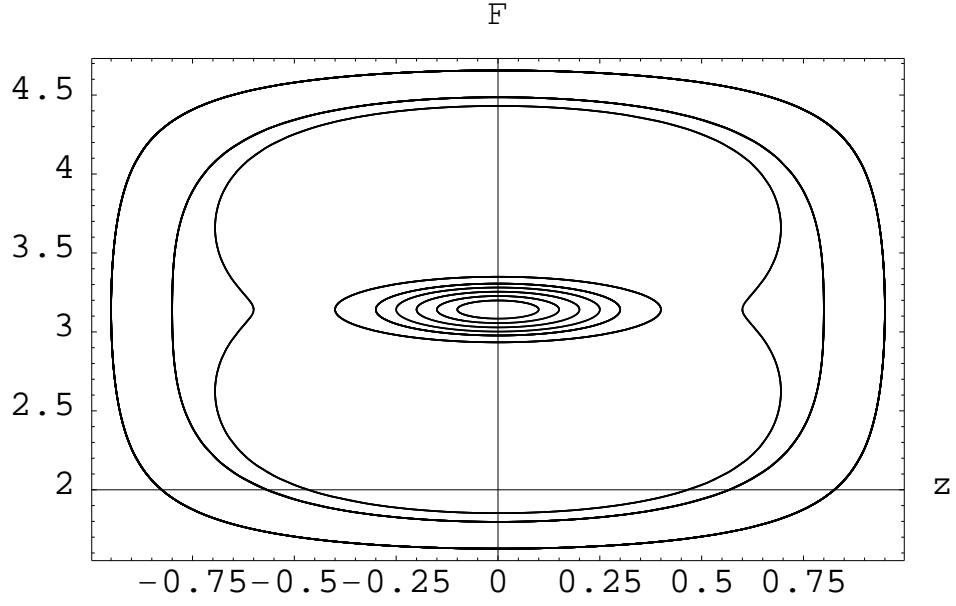
FIG. 11. The phase portrait in a plane z, ϕ of the averaged system of equations (40) for $\Phi(0) = \pi, \Lambda = 2.0, \delta = 0.8$

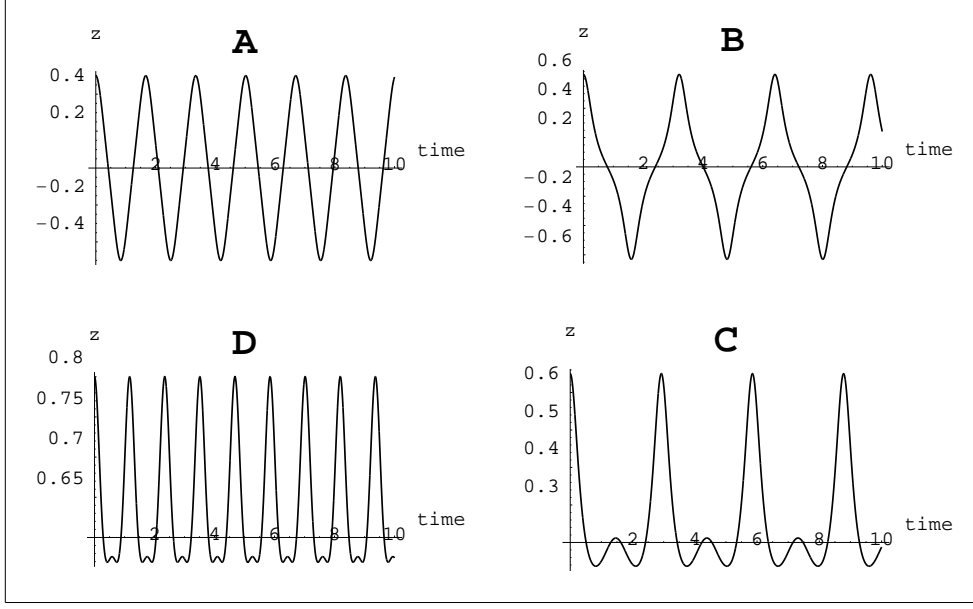FIG. 12. The typical behavior of the solutions to Eq.(2), for $\delta = 0.8$ and $\Lambda = 10, \Phi(0) = 0$ and varying $z(0)$. (**A**) corresponds to $z(0) = 0.5$, (**B**) to $z(0) = 0.65$, (**C**) to $z(0) = 0.68$ and **D** to $z(0) = 0.80$. (**B**) is just before the localization transition and (**C**) just after.